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COMPARISON OF STATIONARY AND QUASI-STATIONARY STREAMS OF PERFECT FLUID

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The nonstationary stream of perfect fluid whose streamlines are stationary (a quasi-stationary stream) is considered in a conservative field of external forces. Conditions under which a field of unit vectors can simultaneously represent the field of velocity directions of a barotropic quasi-stationary potential or vortex motion of a perfect fluid are determined. Comparison is made with the results cited in [1] for a stationary motion, and the absolute values of velocities of stationary and quasi-stationary streams with common streamlines are compared in certain classes of motion. Comparison is also made with the results presented in [2] for a quasi-stationary stream of a perfect incompressible fluid. The range of the considered classes of unit vector fields, the arbitrariness of determination of the absolute value of velocity, and the acceleration and density potentials for a given velocity direction field are determined. It is assumed that the arbitrariness of solutions is determined in a class of analytic functions and that vector fields are analytic.

1. We denote the unit vector of velocity direction by $\mathbf{e}$ and the vector of streamline curvature by $\mathbf{k}$. A vector field is called holonomic if there exists a set of surfaces orthogonal to it [1]. The quantity $H = \operatorname{div} \mathbf{e}$ is called the mean curvature of field $\mathbf{e}$ [3]. It is assumed that some of the vector lines of the field are not straight.

Any holonomic field $\mathbf{e}$ may be considered to be the velocity direction field of a stationary stream of perfect fluid [4]. For a potential quasi-stationary stream of perfect fluid the statement formulated in Theorem 2 in [2] for such fluid is valid. The geometry of velocity directions of such fields for a perfect incompressible fluid is different, since the incompressibility imposes an additional condition on the velocity direction field (condition 2 in Theorem 1 in [2]). For the quasi-stationary stream of perfect fluid we have the following theorems.

Theorem 1. The absolute values of velocities W and V of a stationary and quasi-stationary streams with common streamlines are related by the expression $V = \psi W$, where ψ is a function which at every instant satisfies the condition $\mathbf{e} \times \operatorname{grad} \psi = 0$.

Since for a perfect incompressible fluid ψ depends only on time (note to Theorem 2 in [2]), hence the arbitrariness of determination of the absolute value of velocity for a given velocity direction field is different.

Theorem 2. The holonomic field whose field of curvature vectors is nonholonomic can represent the velocity direction field only for a potential quasi-stationary stream. If fields e and k are holonomic, there exists, also, a quasi-stationary vortex stream whose velocity direction field is e and rot V is parallel to $\mathbf{k} \times \mathbf{e}$. The class of holonomic unit vector fields whose curvature vector field is holonomic, consists of three functions of two arguments. Two functions of two arguments represent the arbitrariness of determination of the absolute value of velocity for a given field e.

It follows from these theorems and [1] that the conditions which are satisfied by velocity direction fields of potential and vortex quasi-stationary streams with common streamlines are the same as for a stationary stream. The class of velocity direction fields common to potential and vortex quasi-stationary streams of incompressible fluid is narrower (the second and fourth conditions of Theorem 3 in [2] are added).

Let us consider the class of motions of a perfect fluid for which the derivative of the absolute value of velocity in the direction of velocity is zero. A characteristic of such motion is that at any instant the absolute value of velocity of all particles of fluid along one and the same arbitrary streamline is the same, and for a stationary motion it is constant. For brevity we call such motions Class I motions.

Theorem 3. A field of unit vectors e can represent the general field of velocity directions of stationary and quasi-stationary streams of Class I then and only then, when it is holonomic and its curvature vector field is potential. If U and W are the absolute values of velocities of Class I stationary and potential stationary streams with the same streamlines, respectively, then function $V = \psi(t) W + U$, where $\psi(t)$ is an arbitrary function of time, can be taken as the absolute value of velocity V of any quasi-stationary stream of Class I with the same streamlines. Conversely, the absolute value of velocity V of any quasi-stationary stream of Class I can be represented in the form $V = \psi(t) W + U$, where $\psi(t)$ is a certain function of time, and U and W are, respectively, the absolute values of velocities of some stationary and potential Class I streams with the same streamlines. The class of holonomic unit vector fields whose curvature vector field is potential consists of two functions of two arguments.

The relationship between the absolute values of velocities in the case of Class I motions of incompressible fluid is the same as above (Theorem 4 in [2]), while the range of direction fields becomes narrower, since the condition of incompressibility introduces the additional requirement for the mean curvature of the velocity direction field to vanish.

Let us consider the motions of a perfect fluid in which the total acceleration is orthogonal to velocity. For such motions the absolute value of velocity of every fluid particle is constant.

Theorem 4. The velocity direction field of a quasi-stationary stream whose total acceleration is orthogonal to velocity can only be represented (except a rectilinear field) by field e which satisfies conditions:

1) the curvature vector field k of field e is holonomic, and
2) the following equality holds:

$$\mathbf{k} \times \{2 \operatorname{grad} \ln [(\mathbf{e} \times \mathbf{k} \cdot \operatorname{rot} \mathbf{k}) |\mathbf{k}|^{-2}] - (\mathbf{k} \times \operatorname{rot} \mathbf{k}) |\mathbf{k}|^{-2}\} = 0 \qquad (1.1)$$

The class of such fields comprises four functions of two arguments.

2. The above results were obtained by analyzing the compatibility of the system of differential equations of hydrodynamics by the method of Kartan.

Let a nonrectilinear field of unit vectors $\mathbf{e}$ be specified in some three-dimensional region of a three-dimensional Euclidean space, which is equivalent to specifying the congruence of lines. As in [2] we adjoin to every point which specifies the field the Frenet's reference point of the field vector line passing through that point. Let $\mathbf{e}_3 = \mathbf{e}, \mathbf{e}_2, \mathbf{e}_1$ be unit vectors of the tangent, the principal normal, and the binormal, respectively, and $\mathbf{M}$ be the radius vector of that point. The differential forms ω_j^i and coefficients a_i, b_i and c_i have the same meaning as in [1, 2]. Then $\mathbf{k} = a_3\mathbf{e}_2$.

Let $\mathbf{V} = V\mathbf{e}_3$ be the velocity of a perfect fluid stream. The system of hydrodynamic equations can now be written in the form of the following system of complete differentials:

$$d\rho = \rho_1\omega^1 + \rho_2\omega^2 + \rho_3\omega^3 - \{\rho_3 V + \rho V_3 + \rho V H\}\, dt \tag{2.1}$$

$$dV = V_1\omega^1 + V_2\omega^2 + V_3\omega^3 + V_t dt \tag{2.2}$$

$$d\varphi = V^2 a_3\omega^2 + (V_t + VV_3)\,\omega^3 + \varphi_t dt \tag{2.3}$$

where the first equation is that of continuity and the third is Euler's equation (φ is the acceleration potential and ρ is the density). The second equality is an expansion of dV in terms of basic forms $\omega^i = d\mathbf{M}\cdot\mathbf{e}_i$ and dt.

The investigation of system (2.1) –(2.3) for compatibility can be reduced to the investigation of its subsystem (2.2), (2.3), since for any fixed solution of the latter, Eq. (2.1) is in involution with respect to ρ, ρ_1, ρ_2 and ρ_3. One arbitrary function of three arguments determines its existence. Because of this we investigate below the system (2.2), (2.3).

Let us consider the potential quasi-stationary stream of perfect fluid. Since $\operatorname{rot}\mathbf{V} = (V_2 - a_3 V)\,\mathbf{e}_1 - V_1\mathbf{e}_2 + V(a_1 - b_2)\,\mathbf{e}_3$, hence for a potential stream the system (2.2), (2.3) is of the form

$$dV = Va_3\omega^2 + V_3\omega^3 + V_t dt \tag{2.4}$$

$$d\varphi = V^2 a_3\omega^2 + (V_t + VV_3)\,\omega^3 + \varphi_t dt \tag{2.5}$$

$$V_1 = 0, \qquad V_2 = Va_3, \qquad a_1 - b_2 = 0 \qquad (\operatorname{rot}\mathbf{V} = 0)$$

The condition $a_1 - b_2 = 0$ implies that the velocity direction field of a potential quasi-stationary stream is holonomic [1]. Conversely, let the holonomic field of unit vectors $\mathbf{e}$ be specified. Differentiating externally system (2.4), (2.5), we obtain a quadratic system which is in involution with respect to V, V_3, V_t and φ_t. This can be proved by constructing a regular chain of solutions by the method if Keler [5]. One arbitrary function of two arguments determines the existence of solution of system (2.4), (2.5). The absolute value of velocity V is determined by Eq. (2.4) with one arbitrary function of two arguments, while φ for any fixed V is determined by one arbitrary function of one argument.

Thus any holonomic field $\mathbf{e}$ can represent the velocity direction field of a potential quasi-stationary stream of perfect fluid. It can, also, represent the velocity direction field of a potential stationary motion [4] the absolute value of whose velocity is determined by an arbitrary function of one argument.

Let a holonomic field $\mathbf{e}$ be specified. The absolute values of velocities W and V of the stationary and quasi-stationary streams associated with that velocity direction field are determined by equations

$$dW = Wa_3\omega^2 + W_3\omega^3 \tag{2.6}$$

and (2.4), respectively. Let $d\psi = \psi_3\omega^3 + \psi_t dt$ and W be any solution of (2.6). Then $W\psi$ satisfies Eq. (2.4). Conversely, if W and V are solutions of Eqs. (2.6) and (2.4), then $(VW^{-1})_1 = (VW^{-1})_2 = 0$. Theorem 1 is proved.

Let us consider the stream of perfect fluid whose velocity direction field is holonomic. It follows from Eq. (2.3) and the condition $a_1 - b_2 = 0$ of the field holonomy that $V_1 = 0$. Hence rot **V** is parallel to $\mathbf{e}_1$ and $V(a_1 - c_3)(V_2 - a_3V) = 0$. This implies that $V_2 = Va_3$, when $a_1 - c_3 \neq 0$, i.e. the stream is potential. Let now $a_1 - c_3 = 0$, hence field **k** is holonomic, and it follows from system (2.2), (2.3) that

$$V_{t2} + VV_{23} + a_2VV_2 + V_2V_3 - a_3V_t - V^2(a_{33} + a_2a_3) - 2a_3VV_3 = 0$$

Taking this formula into consideration and extending Eq. (2.2) to $V_1 = 0$, we obtain

$$\begin{aligned} dV_2 &= -c_2V_2\omega^1 + V_{22}\omega^2 + V_{23}\omega^3 + \{-VV_{23} - a_2VV_2 - V_2V_3 + \\ &\quad a_3V_t + V^2(a_{33} + a_2a_3) + 2a_3VV_3\}\,dt \\ dV_3 &= (V_{23} + a_2V_2 + a_3V_3)\,\omega^2 + V_{33}\omega^3 + V_{t3}dt \\ dV_t &= \{-VV_{23} - a_2VV_2 - V_2V_3 + a_3V_t + \\ &\quad V^2(a_{33} + a_2a_3) + 2a_3VV_3\}\,\omega^2 + V_{t3}\omega^3 + V_{tt}\,dt \end{aligned} \qquad (2.7)$$

Differentiating externally and substituting variables

$$dV_{22} = L^1,\ dV_{23} = L^2,\ dV_{33} = L^3,\ dV_{t3} + VdV_{33} = L^4$$
$$dV_{tt} + VdV_{t3} + V(dV_{t3} + VdV_{33}) = L^5,\ \omega^1 = \omega_0^1$$
$$\omega^2 = \omega_0^2 + dt,\ \omega^3 = \omega_0^3 + Vdt,\ dt = dt_0$$

we obtain a system in involution the existence of whose solution is determined by two arbitrary functions of two arguments. System (2.7) defines V with that arbitrariness, while function φ for the obtained V is determined by an arbitrary function of one argument.

Let us determine the range of the class of holonomic fields of unit vectors **e**, whose curvature vector field is holonomic. Since $a_1 - b_2 = 0$ and $a_1 - c_3 = 0$, hence

$$\omega_3^2 = a_1\omega^1 + a_2\omega^2 + a_3\omega^3, \qquad \omega_3^1 = b_1\omega^1 + a_1\omega^2$$
$$\omega_1^2 = c_1\omega^1 + c_2\omega^2 + a_1\omega^3$$

External differentiation yields a system in involution the existence of whose solution is determined by three arbitrary functions of two arguments. Theorem 2 is proved.

Let us prove Theorem 3. According to condition $V_3 = 0$. It is shown in [1] that the velocity direction field $W\mathbf{e}_3$ of a stationary stream with $W_3 = 0$ satisfies the condition rot $\mathbf{k} \times \mathbf{e} = 0$, i.e. $a_1 - c_3 = 0$ and $a_{33} + a_2a_3 = 0$. Can such field **e** represent also the velocity direction field of a quasi-stationary stream with $V_3 = 0$? Let $a_1 - c_3 = 0$, $a_{33} + a_2a_3 = 0$ and $V_3 = 0$. The system (2.2), (2.3) then assumes the form

$$dV = V_1\omega^1 + V_2\omega^2 + V_tdt, \quad d\varphi = V^2a_3\omega^2 + V_t\omega^3 + \varphi_tdt \qquad (2.8)$$

Let us consider the case of a nonholonomic field **e**, i.e. such that $a_1 - b_2 \neq 0$. After external differentiation and solution by Kartan's lemma, from (2.8) we obtain the relationships

$$V_{t1} = V_{1t} = 0, \qquad V_{t2} = a_3V_t, \qquad V_{t3} = 0$$
$$V_1 = -(2a_3)^{-1}\{(a_{31} + c_2a_3)V + (a_1 - b_2)V^{-1}V_t\}$$
$$V_{tt} = V_t^2V^{-1} - (a_{31} + c_2a_3)(a_1 - b_2)^{-1}VV_t$$

Hence

$$dV_t = V_ta_3\omega^2 + \{V_t^2V^{-1} - (a_{31} + c_2a_3)(a_1 - b_2)^{-1}VV_t\}dt$$

This implies that $(a_{31} + c_2 a_3)\, V_t = 0$, hence $a_{31} + c_2 a_3 = 0$. In this case $dV_t = V_t a_3 \omega^2 + V_t^2 V^{-1} dt$, and external differentiation of this equation and its solution by Kartan's lemma yields $(a_1 - b_2)\, V_t = 0$, i.e. $V_t = 0$. This proves that a quasi-stationary stream of Class I with such velocity direction field is not possible.

Let now field **e** satisfy conditions

$$(a_1 - b_2) = 0, \quad a_1 - c_3 = 0, \; a_{33} + a_2 a_3 = 0 \tag{2.9}$$

By extending system (2.8) we obtain a system in involution the existence of whose solution is determined by three functions of one argument and V is determined by two functions of one argument. Thus only fields which satisfy conditions (2.9), i.e. holonomic fields **e** whose curvature vector field is potential, can represent a velocity direction field which is common to stationary and quasi-stationary stream of Class I. The relationship between the absolute values of velocities of stationary and quasi-stationary streams with common streamlines is derived in exactly the same manner as in Theorem 4 in [2]. Let us determine the range of the class of holonomic vector fields whose curvature vector field is potential. For that class of fields we have

$$\begin{aligned} &da_1 = a_{11}\omega^1 + a_{12}\omega^2 - (2a_1 b_1 + c_2 a_3)\,\omega^3, \quad da_2 = (a_{12} - 2a_1 c_1 - a_2 c_2 + \\ &\quad b_1 c_2)\,\omega^1 + a_{22}\omega^2 + (a_{32} - a_2^2 - a_3^2 - 3a_1^2)\omega^3 \\ &da_3 = -c_2 a_3 \omega^1 + a_{32}\omega^2 - a_2 a_3 \omega^3, \quad db_1 = b_{11}\omega^1 + (a_{11} + \\ &\quad 2a_1 c_2 + b_1 c_1 - a_2 c_1)\,\omega^2 + (a_1^2 - b_1^2 - a_3 c_1)\,\omega^3, \quad dc_1 = \\ &\quad c_{11}\omega^1 + c_{12}\omega^2 + (a_{11} - b_1 c_1 + a_2 b_1)\,\omega^3, \quad dc_2 = (c_{12} + a_1^2 - c_1^2 - c_2^2 - \\ &\quad b_1 a_2)\,\omega^1 + c_{22}\omega^2 + (a_{12} - 2a_1 c_1 - a_2 c_2)\,\omega^3 \end{aligned}$$

External differentiation yields a quadratic system in involution. By constructing a regular chain of solutions we find that the range consists of two functions of two arguments. All statements of Theorem 3 are thus proved.

Let us consider quasi-stationary streams whose total acceleration is orthogonal to velocity, i.e. $V_t + VV_3 = 0$. In this case from (2.2) and (2.3) we obtain

$$\begin{aligned} a_1 - c_3 = 0, \qquad V_1 = -(2a_3)^{-1}(a_{31} + c_2 a_3)\, V \\ V_3 = -(2a_3)^{-1}(a_{33} + a_2 a_3)\, V \end{aligned}$$

The relationship

$$\begin{aligned} dV = -(2a_3)^{-1}(a_{31} + c_2 a_3) V\omega^1 + V_2\omega^2 - (2a_3)^{-1}(a_{33} + a_2 a_3)\, V\omega^3 + \\ (2a_3)^{-1}(a_{33} + a_2 a_3) V^2 dt \end{aligned}$$

implies that the velocity direction field of such stream must satisfy relationships

$$\begin{aligned} \{a_3^{-1}(a_{33} + a_2 a_3)\}_1 = (2a_3^2)^{-1}(a_{31} + c_2 a_3)(a_{33} + a_2 a_3) \\ \{a_3^{-1}(a_{33} + a_2 a_3)\}_3 = (2a_3^2)^{-1}(a_{33} + a_2 a_3)^2, \; a_1 - c_3 = 0 \end{aligned} \tag{2.10}$$

If $a_{33} + a_2 a_3 = 0$, the stream is stationary, and conversely, if the stream is stationary, $a_1 - c_3 = 0$ and $a_{33} + a_2 a_3 = 0$ (see [1]). For $a_{33} + a_2 a_3 \neq 0$ conditions (2.10) can be represented in the form of equality (1.1) and $\mathbf{k} \cdot \mathrm{rot}\, \mathbf{k} = 0$. In this case

$$V = -2a_3 (a_{33} + a_2 a_3)^{-1} (t + N)^{-1}$$

where N is any function which satisfies the condition $dN = N_2 \omega^2$. The class of fields which satisfy conditions (2.10) contains four functions of two arguments. This can be proved by constructing the regular chain of solutions for the system which defines this class of fields. Theorem 4 is proved.

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STEADY FLOW OF A VISCOUS INCOMPRESSIBLE FLUID TAKING TEMPERATURE DEPENDENCE OF VISCOSITY INTO ACCOUNT

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We consider the two-dimensional steady flows of a viscous incompressible fluid with viscosity depending exponentially on the temperature. In contrast to the numerical methods for solving this problem [1], we reduce the nonlinear system of equations describing the flow to an infinite sequence of linear equations of elliptic type by means of an expansion in the small parameter appearing in the exponent. We construct a majorizing equation for which the existence of positive solutions guarantees the uniform convergence of iterations on a neighborhood of a zero value of the parameter. As an illustration we study the flow of a viscous fluid in a cylindrical tube with a heat source present.

1. Consider the steady two-dimensional flow of a viscous incompressible fluid with the temperature-dependent viscosity given by the Reynolds relation

$$\mu/\mu_0 = e^{-\alpha T}$$

The system of differential equations of motion, continuity, and energy has the following form [2] upon the introduction of a stream function and omission of the inertial and dissipative terms:

$$\Delta\Delta\psi = 2\alpha\left(\frac{\partial T}{\partial x}\frac{\partial}{\partial x}\Delta\psi + \frac{\partial T}{\partial y}\frac{\partial}{\partial y}\Delta\psi\right) - \left[\alpha^2\left(\frac{\partial T^2}{\partial y} - \frac{\partial T^2}{\partial x}\right) - \alpha\left(\frac{\partial^2 T}{\partial y^2} - \frac{\partial^2 T}{\partial x^2}\right)\right]\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) - 4\left(\alpha^2\frac{\partial T}{\partial x}\frac{\partial T}{\partial y} - \alpha\frac{\partial^2 T}{\partial x\partial y}\right)\frac{\partial^2\psi}{\partial x\partial y} \tag{1.1}$$

$$\Delta T = P\left(\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}\right), \quad \alpha = \beta\delta T, \quad P = \frac{VL}{a}$$

The geometrical and physical flow parameters are assumed to be dimensionless, being referred to characteristic scaling parameters: the length L, the difference of tempera-